

Level sets of the resolvent norm of a linear operator revisited*

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Abstract

It is proved that the resolvent norm of an operator with a compact resolvent on a Banach space X cannot be constant on an open set if the underlying space or its dual is complex strictly convex. It is also shown that this is not the case for an arbitrary Banach space: there exists a separable, reflexive space X and an unbounded, densely defined operator acting in X with a compact resolvent whose norm is constant in a neighbourhood of zero; moreover X is isometric to a Hilbert space on a subspace of co-dimension 2. There is also a bounded linear operator acting on the same space whose resolvent norm is constant in a neighbourhood of zero. It is shown that similar examples cannot exist in the co-dimension 1 case.

1 Introduction

The ε -pseudospectrum of a closed densely defined linear operator A on a Banach space X is usually defined as

$$\sigma_\varepsilon(A) = \{\lambda \in \mathbb{C} : \|(A - \lambda I)^{-1}\| > 1/\varepsilon\} \quad (1)$$

or as

$$\Sigma_\varepsilon(A) = \{\lambda \in \mathbb{C} : \|(A - \lambda I)^{-1}\| \geq 1/\varepsilon\}, \quad (2)$$

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where $\varepsilon > 0$ and $\|(A - \lambda I)^{-1}\|$ is assumed to be infinite if $\lambda \in \sigma(A)$ (see, e.g., [5, 7, 8, 13, 37, 38] and [6, 17]). The difference between $\Sigma_\varepsilon(A)$ and $\sigma_\varepsilon(A)$ is the (closed) level set

$$\{\lambda \in \mathbb{C} : \|(A - \lambda I)^{-1}\| = 1/\varepsilon\} \quad (3)$$

and it is natural to ask whether this set may have an open subset, in which case $\Sigma_\varepsilon(A)$ is strictly larger than the closure of $\sigma_\varepsilon(A)$. If this happens at a point $\varepsilon = \varepsilon_0$, then the ε -pseudospectrum of A jumps as ε passes through ε_0 .

The question on whether or not the level set (3) may have an open subset goes back to J. Globevnik (see [19]) who showed that the resolvent norm of a bounded linear operator on a Banach space cannot be constant on an open set if the underlying space is complex uniformly convex (see Definition A.1 below). An easy duality argument shows that this remains true if the dual of the underlying Banach space, rather than the space itself, is complex uniformly convex. Hence the class of spaces to which Globevnik's result applies includes Hilbert spaces and $L^p(S, \Sigma, \mu)$ with $1 \leq p \leq \infty$, where (S, Σ, μ) is an arbitrary measure space (see [33] or Appendix A below).

An example of a bounded linear operator on a Banach space for which the resolvent norm is constant in a neighbourhood of zero was constructed in [33] and then modified in [35] to make the underlying space separable, reflexive and strictly convex.

According to the above, the resolvent norm of a bounded linear operator on a Hilbert space cannot be constant on an open set, and there have been several claims in the literature that the same is true for a closed densely defined operator on a Hilbert space. A counterexample to those claims was constructed in [33], where it was shown that there exists a block diagonal closed densely defined operator on $\ell_2(\mathbb{N})$ with 2×2 blocks, such that its resolvent norm is constant in a neighbourhood of zero. It is natural to ask whether this phenomenon can occur for “non-pathological” unbounded operators arising in “real” applications.

It was shown in [34] that the answer to this question is negative for semi-group generators: the resolvent norm of the infinitesimal generator of a C_0 semigroup on a Banach space cannot be constant on an open set if the underlying space is complex uniformly convex. This result can also be easily derived with the help of the Hille-Yosida theorem from a recent result by S. Bögli and P. Siegl ([3]) about closed operators acting on a complex uniformly convex Banach space, which says that if the resolvent norm is constant on an open set, then this constant is the global minimum. The examples from [33] and [35] show that one cannot drop the requirement of complex uniform

convexity in these results, relax it to complex strict convexity or even replace it with strict convexity.

Here, we consider another important class of unbounded operators, namely operators with compact resolvents. In Theorem 2.2 we show that the resolvent norm of such an operator cannot be constant on an open set if the underlying Banach space is complex **strictly** convex. So, the situation here is slightly different from what one has for bounded operators. Unlike previous results, Theorem 2.2 is applicable to small perturbations of operators such as $(Af)(m, n) = (m + in)f(m, n)$ acting in $l^2(\mathbb{N} \times \mathbb{N})$; see the case $a = 0$ of [13, Theorem 11.1.3]. This operator has compact resolvent and the resolvent norm is uniformly bounded away from 0 for $\lambda \notin \sigma(A)$. In Theorem 2.3 we show that the example in [33] can be modified to produce an operator on a suitable Banach space \mathcal{X} for which the resolvent is compact and the resolvent norm is constant in a neighbourhood of zero. Perhaps the most interesting part of this result is that \mathcal{X} is isometric to a Hilbert space on a subspace of co-dimension 2, and we show in Theorem 2.4 that the same formula as in the example in [33] defines a bounded linear operator on \mathcal{X} whose resolvent norm is constant in a neighbourhood of zero. Theorems 3.2 and 3.3 show that similar examples cannot exist in the co-dimension 1 case.

We use A to denote a closed densely defined operator with a compact resolvent and B to denote a bounded linear operator or the infinitesimal generator of a C_0 semigroup. We denote by X and Y Banach spaces satisfying certain convexity hypothesis, while the calligraphic letters \mathcal{X} and \mathcal{Y} are used to denote the spaces $l^2(\mathbb{Z}) \oplus \mathbb{C}^2$ and $Y \oplus \mathbb{C}$ equipped with suitable norms. We devote Appendix A to presenting some known results on convexity properties and absolute norms that are used in the paper.

We conclude this introduction by listing the theorems in the order that they appear below and make brief comments about each one; these comments do not pretend to give full descriptions of the conditions in the theorems. The symbol \exists denotes that the theorem proves the existence of an operator in some stated class whose resolvent norm has at least one level set with non-empty interior, while \mathbf{N} denotes that no operator in some stated class possesses a resolvent whose norm has such a level set.

Theorem	Banach space	\exists/N	Comments
2.2	X	N	A has a compact resolvent;
2.3	$\ell^2(\mathbb{Z}) \oplus \mathbb{C}^2$	\exists	A has a compact resolvent;
2.4	$\ell^2(\mathbb{Z}) \oplus \mathbb{C}^2$	\exists	B is bounded;
3.1	$Y \oplus_\infty \mathbb{C}$	\exists	A has a compact resolvent, B is bounded;
3.2	$Y \oplus \mathbb{C}$	N	A has a compact resolvent;
3.3	$Y \oplus \mathbb{C}$	N	B generates a C_0 semigroup.

2 Main results

Some of our results depend on the following classical theorem; see [21, Theorem 2.16.5] or [25, Ch. III, Theorems 5.30 and 6.22].

Theorem 2.1. *Let H be a closed densely defined operator acting in the Banach space X . Then its dual H^* has a weak* dense domain in X^* and its graph is weak* closed. Moreover $\text{Spec}(H) = \text{Spec}(H^*)$ and*

$$\{(H - \lambda I)^{-1}\}^* = (H^* - \lambda I)^{-1} \quad (4)$$

for all $\lambda \notin \text{Spec}(H)$. In particular

$$\|(H - \lambda I)^{-1}\| = \|(H^* - \lambda I)^{-1}\|$$

for all $\lambda \notin \text{Spec}(H)$.

Our next theorem holds when X is a Hilbert space, and appears to be new even in that case.

Theorem 2.2. *Suppose a Banach space X or its dual X^* is complex strictly convex in the sense of Definition A.1, and $A : X \rightarrow X$ is a closed densely defined operator with a compact resolvent $R(\lambda) := (A - \lambda I)^{-1}$. Let Ω be an open subset of the resolvent set of A . If $\|R(\lambda)\| \leq M$ for all $\lambda \in \Omega$, then $\|R(\lambda)\| < M$ for all $\lambda \in \Omega$.*

Proof. The proof is similar to that of [33, Theorem 2.6]. One can assume without loss of generality that Ω is connected. Indeed, it is sufficient to consider each connected component of Ω separately.

Part 1. We consider first the case in which X is complex strictly convex. Suppose that there exists $\lambda_0 \in \Omega$ such that $\|R(\lambda_0)\| = M$. Then [33, Theorem 2.1] or the maximum principle (see, e.g., [21, Theorem 3.13.1] or [16, Ch. III,

Sect. 14]) imply that $\|R(\lambda)\| = M$, $\forall \lambda \in \Omega$. Shifting the independent variable if necessary, we can assume that $0 \in \Omega$.

According to [20, Lemma 1.1], there exists $r > 0$ such that

$$\|R(0) + \lambda R^2(0)\| = \|R(0) + \lambda R'(0)\| \leq M, \quad |\lambda| \leq r.$$

Since $\|R(0)\| = M$, there exist $u_n \in X$, $n \in \mathbb{N}$ such that $\|u_n\| = \frac{1}{M}$ and $\|R(0)u_n\| \rightarrow 1$ as $n \rightarrow \infty$. Since $R(0)$ is compact, one can assume, after going to a subsequence, that $R(0)u_n$ converges to a vector $x \in X$ and $\|x\| = 1$. Then $y := rR(0)x \neq 0$ and

$$\begin{aligned} \|x + \zeta y\| &= \lim_{n \rightarrow \infty} \|R(0)u_n + \zeta r R^2(0)u_n\| \leq \|R(0) + \zeta r R^2(0)\| \|u_n\| \\ &\leq M \frac{1}{M} = 1, \quad |\zeta| \leq 1. \end{aligned}$$

The contradiction implies that there does not exist $\lambda_0 \in \Omega$ such that $\|R(\lambda_0)\| = M$.

Part 2. Let us now consider the case where X^* is complex strictly convex. If X is reflexive, then A^* is a closed, densely defined operator (see, e.g., [21, Theorems 2.11.8 and 2.11.9] or [25, Ch. III, Theorem 5.29]) with a compact resolvent, and our claim follows by duality from what has been proved in Part 1 above. If X is not reflexive, A^* might not be densely defined, but one can repeat the argument by using Theorem 2.1. This implies that the resolvent $R^*(\lambda)$ of A^* is compact and one-one on X^* , and that its range is weak* dense in X^* and independent of λ . One can now proceed as in Part 1. \square

Let us consider the following norm on $l_2(\mathbb{Z})$:

$$\|x\|_* = \max\{\|x'\|_2, |x_1|\} + |x_0|, \quad x = (x_k)_{k \in \mathbb{Z}}, \quad x' = (x_k)_{k \in \mathbb{Z} \setminus \{0,1\}}. \quad (5)$$

It is easy to see that

$$\begin{aligned} \|x\|_* &\leq \|\tilde{x}\|_2 + |x_0| \leq \sqrt{2} \|x\|_2, \quad \tilde{x} = (x_k)_{k \in \mathbb{Z} \setminus \{0\}}, \\ \|x\|_* &\geq \frac{1}{\sqrt{2}} \|\tilde{x}\|_2 + |x_0| \geq \frac{1}{\sqrt{2}} \|x\|_2. \end{aligned}$$

Hence

$$\frac{1}{\sqrt{2}} \|x\|_2 \leq \|x\|_* \leq \sqrt{2} \|x\|_2, \quad \forall x \in l_2(\mathbb{Z}). \quad (6)$$

Moreover $\|x\|_2 = \|x\|_*$ if $x_0 = x_1 = 0$. Letting \mathcal{X} denote the space $l_2(\mathbb{Z})$ equipped with the norm $\|\cdot\|_*$, it follows that \mathcal{X} is reflexive and separable.

Note that $\mathcal{X} = \mathcal{H} \oplus \mathbb{C}^2$ where

$$\mathcal{H} = \{x \in l_2(\mathbb{Z}) : x_0 = x_1 = 0\}$$

is a Hilbert space with the norm induced by $\|\cdot\|_*$.

Theorem 2.3. *There exists a closed, densely defined operator $A : \mathcal{X} \rightarrow \mathcal{X}$ with a compact resolvent such that $\|(A - \lambda I)^{-1}\|$ is constant in a neighbourhood of $\lambda = 0$.*

Proof. The proof is similar to that of Theorem 3.1 in [33]. We suppose throughout that $\delta = \frac{1}{4}$ and that $\lambda \in \mathbb{C}$ is arbitrary subject to

$$|\lambda| \leq \delta. \quad (7)$$

The same calculations are applicable for any smaller positive value of δ .

Part 1. Let A be the weighted shift operator defined by

$$(Ay)_k = \delta^{-|k|} y_{k+1}, \quad k \in \mathbb{Z},$$

where $\text{Dom}(A)$ is the set of all $y \in \mathcal{X}$ for which $Ay \in \mathcal{X}$. Since $\text{Dom}(A)$ contains all sequences with finite support, it is dense in \mathcal{X} . It is clear that $A : \text{Dom}(A) \rightarrow \mathcal{X}$ is invertible and

$$(A^{-1}x)_k = \beta_k x_{k-1}, \quad k \in \mathbb{Z}, \quad (8)$$

for all $x \in \mathcal{X}$, where

$$\beta_k = \delta^{|k-1|}, \quad k \in \mathbb{Z}. \quad (9)$$

The formula $\lim_{k \rightarrow \pm\infty} \beta_k = 0$ implies that $A^{-1} : \mathcal{X} \rightarrow \mathcal{X}$ is a compact operator.

Part 2. If one considers A^{-1} as an operator on $l_2(\mathbb{Z})$ equipped with the standard norm, then it is clear that $\|A^{-1}\| = 1$ (see (8), (9)), and hence

$$(A - \lambda I)^{-1} = \sum_{j=0}^{\infty} \lambda^j A^{-(j+1)}, \quad |\lambda| < 1. \quad (10)$$

Since \mathcal{X} coincides with $l_2(\mathbb{Z})$ as a set and is equipped with a norm equivalent to that of $l_2(\mathbb{Z})$, we conclude that $A - \lambda I : \text{Dom}(A) \rightarrow \mathcal{X}$ is invertible when $|\lambda| < 1$, and hence when λ satisfies (7), and that (10) remains valid in this setting. (Actually, it is not difficult to show that the equality $\|A^{-1}\| = 1$ holds for the operator $A^{-1} : \mathcal{X} \rightarrow \mathcal{X}$. Equality (14) below, which is the main claim of the theorem, is a considerably stronger statement.)

Part 3. Take an arbitrary $x \in \mathcal{X}$ such that $\|x\|_* \leq 1$, and note that this implies $|x_k| \leq 1 - |x_0|$ for all $k \neq 0$. Assume that $\lambda \in \mathbb{C}$ satisfies (7). Let $y = (A - \lambda I)^{-1}x$. Since

$$(A^{-(j+1)}x)_k = \beta_k \beta_{k-1} \cdots \beta_{k-j} x_{k-1-j}$$

for all $k \in \mathbb{Z}$ and all $j \in \mathbb{N} \cup \{0\}$, one has

$$\begin{aligned} |y_0| &= |\beta_0 x_{-1} + \lambda \beta_0 \beta_{-1} x_{-2} + \lambda^2 \beta_0 \beta_{-1} \beta_{-2} x_{-3} + \cdots| \\ &\leq \delta(1 - |x_0|)(1 + |\lambda| + |\lambda|^2 + \cdots) \\ &= \delta \frac{1 - |x_0|}{1 - |\lambda|}, \end{aligned}$$

and

$$\begin{aligned} |y_1| &= |\beta_1 x_0 + \lambda \beta_1 \beta_0 x_{-1} + \lambda^2 \beta_1 \beta_0 \beta_{-1} x_{-2} + \cdots| \\ &\leq |x_0| + \delta(1 - |x_0|)(|\lambda| + |\lambda|^2 + \cdots) \\ &= |x_0| + \delta |\lambda| \frac{1 - |x_0|}{1 - |\lambda|}. \end{aligned}$$

Combining these bounds yields

$$\begin{aligned} |y_1| + |y_0| &\leq |x_0| + \delta |\lambda| \frac{1 - |x_0|}{1 - |\lambda|} + \delta \frac{1 - |x_0|}{1 - |\lambda|} \\ &= |x_0| + \delta \frac{1 + |\lambda|}{1 - |\lambda|} (1 - |x_0|) \\ &\leq |x_0| + (1 - |x_0|) \\ &= 1. \end{aligned} \tag{11}$$

Part 4. We use the fact that $\|x\|_* \leq 1$ implies $|x_k| \leq 1$ for all $k \in \mathbb{Z}$. Then

$$\begin{aligned} |y_k| &= |\beta_k x_{k-1} + \lambda \beta_k \beta_{k-1} x_{k-2} + \lambda^2 \beta_k \beta_{k-1} \beta_{k-2} x_{k-3} + \cdots| \\ &\leq \delta^{|k-1|} (1 + |\lambda| + |\lambda|^2 + \cdots) \\ &= \frac{\delta^{|k-1|}}{1 - |\lambda|} \leq \frac{4}{3} \delta^{|k-1|}. \end{aligned}$$

In particular $|y_0| \leq \frac{1}{3}$. Hence

$$\begin{aligned} \left(\sum_{k \neq 0,1} |y_k|^2 \right)^{1/2} + |y_0| &\leq \left(\left(\frac{4}{3} \right)^2 \sum_{k \leq -1} \delta^{2(1-k)} + \left(\frac{4}{3} \right)^2 \sum_{k \geq 2} \delta^{2(k-1)} \right)^{1/2} + \frac{1}{3} \\ &\leq \frac{4}{3} \left(\frac{\delta^4}{1 - \delta^2} + \frac{\delta^2}{1 - \delta^2} \right)^{1/2} + \frac{1}{3} = \frac{4\delta}{3} \left(\frac{1 + \delta^2}{1 - \delta^2} \right)^{1/2} + \frac{1}{3} < 1. \end{aligned} \tag{12}$$

Part 5. By combining the bounds (11) and (12) we obtain $\|y\|_* \leq 1$ and hence

$$\|(A - \lambda I)^{-1}\| \leq 1. \quad (13)$$

On the other hand, let $z = (A - \lambda I)^{-1}e_0$, where $e_0 := (\dots, 0, 0, 1, 0, 0, \dots)$ with 1 at the 0th place. Then $\|e_0\|_* = 1$,

$$z_k = \begin{cases} 0, & k \leq 0, \\ \beta_1, & k = 1, \\ \beta_1\beta_2 \cdots \beta_k \lambda^{k-1}, & k \geq 2, \end{cases}$$

and

$$\|(A - \lambda I)^{-1}e_0\|_* = \|z\|_* \geq \beta_1 = 1.$$

By combining this with (13), we finally deduce that

$$\|(A - \lambda I)^{-1}\| = 1 \quad (14)$$

under the condition (7). \square

Theorem 2.4. *Let \mathcal{X} denote the space $l_2(\mathbb{Z})$ equipped with the norm $\|\cdot\|_*$ defined in (5). Then there exists an invertible bounded linear operator $B : \mathcal{X} \rightarrow \mathcal{X}$ such that $\|(B - \lambda I)^{-1}\|$ is constant in a neighbourhood of $\lambda = 0$.*

Proof. Part 1. Let $M > 3$ and let B be the weighted shift operator

$$(Bx)_k = \alpha_k x_{k+1}, \quad k \in \mathbb{Z},$$

where

$$\alpha_k = \begin{cases} \frac{1}{M}, & k = 0, \\ 1, & k \neq 0. \end{cases}$$

It is clear that B is invertible on \mathcal{X} and that

$$(B^{-1}y)_k = \beta_k y_{k-1}, \quad k \in \mathbb{Z},$$

where

$$\beta_k = \frac{1}{\alpha_{k-1}} = \begin{cases} M, & k = 1, \\ 1, & k \neq 1. \end{cases}$$

As in Part 2 of the proof of Theorem 2.3, one can consider B^{-1} as an operator on $l_2(\mathbb{Z})$ equipped with the standard norm. Then it is clear that $\|B^{-1}\| = M$, and hence

$$(B - \lambda I)^{-1} = \sum_{j=0}^{\infty} \lambda^j B^{-(j+1)}, \quad |\lambda| < \frac{1}{M}. \quad (15)$$

Again, since \mathcal{X} coincides with $l_2(\mathbb{Z})$ as a set and is equipped with a norm equivalent to that of $l_2(\mathbb{Z})$, we conclude that $B - \lambda I : \mathcal{X} \rightarrow \mathcal{X}$ is invertible when $|\lambda| < \frac{1}{M}$, and that (15) remains valid in this setting. (It is not difficult to show that the equality $\|B^{-1}\| = \frac{1}{M}$ holds for the operator $B^{-1} : \mathcal{X} \rightarrow \mathcal{X}$.)

Part 2. Take an arbitrary $x \in \mathcal{X}$ such that $\|x\|_* \leq 1$ and an arbitrary $\lambda \in \mathbb{C}$ such that $|\lambda| < \frac{1}{M}$. Let $y = (B - \lambda I)^{-1}x$. Since $(B^{-(j+1)}x)_k = \beta_k \cdots \beta_{k-j} x_{k-1-j}$, $j \in \mathbb{N} \cup \{0\}$, we get

$$y_k = \begin{cases} \sum_{j=0}^{\infty} \lambda^j x_{k-1-j}, & k \leq 0, \\ Mx_0 + M \sum_{j=1}^{\infty} \lambda^j x_{-j}, & k = 1, \\ \sum_{j=0}^{k-2} \lambda^j x_{k-1-j} + M\lambda^{k-1}x_0 + M \sum_{j=k}^{\infty} \lambda^j x_{k-1-j}, & k \geq 2. \end{cases} \quad (16)$$

In this part we obtain upper bounds on y by using the decomposition $y' = v' + w'$, where y' is defined as in (5) and

$$v_k = \sum_{j=0}^{\infty} \lambda^j x_{k-1-j} = \sum_{l=1}^{\infty} \lambda^{l-1} x_{k-l}, \quad k \in \mathbb{Z} \setminus \{0, 1\},$$

$$w_k = \begin{cases} 0, & k < 0, \\ (M-1) \sum_{j=k-1}^{\infty} \lambda^j x_{k-1-j}, & k \geq 2. \end{cases}$$

Using the notation

$$\nu = (\nu_l)_{l \in \mathbb{Z}}, \quad \nu_l = \begin{cases} 0, & l < 1, \\ \lambda^{l-1}, & l \geq 1, \end{cases}$$

we get

$$\|v'\|_2 \leq \|\nu * x\|_2 \leq \|\nu\|_1 \|x\|_2 = \frac{1}{1-|\lambda|} \|x\|_2 \leq \frac{\sqrt{2}}{1-|\lambda|} \|x\|_* = \frac{\sqrt{2}}{1-|\lambda|}$$

(see (6)). Further,

$$\begin{aligned} |w_k| &\leq (M-1) \left(\sum_{j=k-1}^{\infty} |\lambda|^{2j} \right)^{1/2} \|x\|_2 = \frac{(M-1)|\lambda|^{k-1}}{(1-|\lambda|^2)^{1/2}} \|x\|_2 \\ &\leq \frac{\sqrt{2}(M-1)|\lambda|^{k-1}}{(1-|\lambda|^2)^{1/2}}, \quad k \geq 2, \\ \|w'\|_2 &\leq \frac{\sqrt{2}(M-1)}{(1-|\lambda|^2)^{1/2}} \left(\sum_{k=2}^{\infty} |\lambda|^{2(k-1)} \right)^{1/2} = \frac{\sqrt{2}(M-1)|\lambda|}{(1-|\lambda|^2)}. \end{aligned}$$

Hence,

$$\begin{aligned}\|y'\|_2 &\leq \|v'\|_2 + \|w'\|_2 \leq \frac{\sqrt{2}}{1-|\lambda|} + \frac{\sqrt{2}(M-1)|\lambda|}{(1-|\lambda|^2)} \\ &= \frac{\sqrt{2}(M|\lambda|+1)}{(1-|\lambda|^2)}.\end{aligned}\quad (17)$$

Since $\|x\|_* \leq 1$ implies $|x_k| \leq 1 - |x_0|$, $k \neq 0$, we may use (16) directly to obtain

$$|y_0| \leq (1 - |x_0|) \sum_{j=0}^{\infty} |\lambda|^j = \frac{1 - |x_0|}{1 - |\lambda|}, \quad (18)$$

$$|y_1| \leq M|x_0| + M(1 - |x_0|) \sum_{j=1}^{\infty} |\lambda|^j = M|x_0| + M|\lambda| \frac{1 - |x_0|}{1 - |\lambda|}. \quad (19)$$

Part 3. Suppose additionally that $|\lambda| < \frac{1}{3} - \frac{1}{M}$. By using (17), (18) and (19) we obtain

$$\begin{aligned}\|y'\|_2 + |y_0| &\leq \frac{\sqrt{2}(M|\lambda|+1)}{(1-|\lambda|^2)} + \frac{1 - |x_0|}{1 - |\lambda|} < \frac{2(M|\lambda|+1)}{1 - |\lambda|} + \frac{1}{1 - |\lambda|} \\ &= \frac{2(M|\lambda|+1)+1}{1 - |\lambda|} < \frac{2\frac{M}{3}+1}{\frac{2}{3} + \frac{1}{M}} = \frac{2M+3}{2M+3} M = M\end{aligned}$$

and

$$\begin{aligned}|y_1| + |y_0| &\leq M|x_0| + (1 - |x_0|) \frac{1 + M|\lambda|}{1 - |\lambda|} \leq M|x_0| + \frac{\frac{M}{3}}{\frac{2}{3} + \frac{1}{M}} (1 - |x_0|) \\ &= M|x_0| + \frac{M}{2M+3} M(1 - |x_0|) \leq M.\end{aligned}$$

Therefore $\|y\|_* \leq M$ and

$$\|(B - \lambda I)^{-1}\| \leq M, \quad |\lambda| < \min \left\{ \frac{1}{M}, \frac{1}{3} - \frac{1}{M} \right\}. \quad (20)$$

Part 4. The proof is completed by combining (20) with a corresponding, but simpler, lower bound. If $z = (B - \lambda I)^{-1}e_0$, where e_0 is the same as in Part 5 of the proof of Theorem 2.3, then

$$z_k = \begin{cases} 0, & k \leq 0, \\ M, & k = 1, \\ M\lambda^{k-1}, & k \geq 2, \end{cases}$$

and $\|z\|_* = \|(B - \lambda I)^{-1}e_0\|_* \geq M$, $|\lambda| < \frac{1}{M}$. Therefore

$$\|(B - \lambda I)^{-1}\| \geq M, \quad |\lambda| < \min \left\{ \frac{1}{M}, \frac{1}{3} - \frac{1}{M} \right\}.$$

□

3 Co-dimension one

We have shown above that there exist a bounded operator and a closed densely defined operator with a compact resolvent on $\mathcal{X} = (l_2(\mathbb{Z}) \oplus_\infty \mathbb{C}) \oplus_1 \mathbb{C}$ whose resolvent norms are constant in a neighbourhood of 0. The norm in X coincides with the l_2 norm on a subspace of co-dimension two, and it is natural to ask whether similar examples exist in co-dimension one.

Theorem 3.1. *There exist a separable, complex uniformly convex Banach space Y , a bounded linear operator B and a closed densely defined operator A with a compact resolvent on $\mathcal{Y} := Y \oplus_\infty \mathbb{C}$ whose resolvent norms are constant in a neighbourhood of 0.*

Proof. Let $c_0(\mathbb{Z})$ denote as usual the subspace of $l_\infty(\mathbb{Z})$ consisting of all elements $(x_k)_{k \in \mathbb{Z}}$ such that $\lim_{|k| \rightarrow \infty} x_k = 0$, and let B_1 be the operator obtained from the one in [33, Theorem 3.1] if one replaces $l_\infty(\mathbb{Z})$ by $Y_1 := c_0(\mathbb{Z})$. The proof of that theorem carries over to B_1 without change. Therefore the resolvent norm of $B_1 : Y_1 \oplus_1 \mathbb{C} \rightarrow Y_1 \oplus_1 \mathbb{C}$ is constant in a neighbourhood of 0. The dual space $Y := l_1(\mathbb{Z})$ of $c_0(\mathbb{Z})$ is complex uniformly convex (see [18]), and the resolvent norm of the adjoint operator $B := B_1^* : Y \oplus_\infty \mathbb{C} \rightarrow Y \oplus_\infty \mathbb{C}$ is constant in a neighbourhood of 0.

Similarly, one can define an operator A_1 by the same formula as in the proof of Theorem 2.3, but replacing $l_2(\mathbb{Z})$ by $Y_1 := c_0(\mathbb{Z})$ there. Then an argument similar to, but easier than, the proof of that theorem shows that $A_1 : Y_1 \oplus_1 \mathbb{C} \rightarrow Y_1 \oplus_1 \mathbb{C}$ is a closed densely defined operator with a compact resolvent whose norm is constant in a neighbourhood of 0. It follows that $A := A_1^* : Y \oplus_\infty \mathbb{C} \rightarrow Y \oplus_\infty \mathbb{C}$ has the desired properties. □

Our next task is to show that one cannot take Y to be a Hilbert space in the above theorem: the resolvent norm of a bounded linear operator or of a closed densely defined operator with a compact resolvent on $l_2 \oplus_p \mathbb{C}$, $1 \leq p \leq \infty$ cannot be constant on an open set. In fact, we prove a more general result which uses absolute norms on \mathbb{C}^2 . Readers not familiar with such norms and with the definitions of Ψ and \oplus_ψ where $\psi \in \Psi$ should refer to Appendix A.

Theorem 3.2. *Let Ω be an open subset of \mathbb{C} and let Y be a complex strictly convex Banach space with a complex strictly convex dual Y^* . Given $\psi \in \Psi$, let $\|\cdot\|$ be the (ψ -dependent) norm on the Banach space $\mathcal{Y} := Y \oplus_{\psi} \mathbb{C}$. Suppose $A : \mathcal{Y} \rightarrow \mathcal{Y}$ is a closed densely defined operator with a compact resolvent $(A - \lambda I)^{-1}$ defined for all $\lambda \in \Omega$. If $\|(A - \lambda I)^{-1}\| \leq M$ for all $\lambda \in \Omega$, then $\|(A - \lambda I)^{-1}\| < M$ for all $\lambda \in \Omega$.*

Proof. One can assume as in the proof of Theorem 2.2 that Ω is connected.

Part 1. If ψ satisfies (32), then \mathcal{Y} is complex strictly convex and our claim follows from Theorem 2.2.

Part 2. Suppose there exist $t_0 \in (0, 1/2]$ and $t_1 \in [1/2, 1)$ such that $\psi(t_0) = 1 - t_0$ and $\psi(t_1) = t_1$. Then ψ^* satisfies (32) (see (33)–(34)), and $Y^* \oplus_{\psi^*} \mathbb{C}$ is complex strictly convex. Since $\mathcal{Y}^* = Y^* \oplus_{\psi^*} \mathbb{C}$ (see [27]), our claim again follows from Theorem 2.2.

Part 3. Suppose $\psi(t) > t$ for all $t \in [1/2, 1)$ and $\psi(t) = 1 - t$ for sufficiently small $t > 0$. Then there exists $t_0 \in (0, 1/2)$ such that $\psi(t) = 1 - t$ for all $t \in [0, t_0]$ and $\psi(t) > 1 - t$ for all $t \in (t_0, 1/2]$.

Suppose there exists $\lambda_0 \in \Omega$ such that $\|R(\lambda_0)\| = M$, where $R(\lambda) := (A - \lambda I)^{-1}$. Then, exactly as in the proof of Theorem 2.2, one can assume that $0 \in \Omega$ and derive from [20, Lemma 1.1] the existence of $r > 0$ such that $\|R(0) + \lambda R'(0)\| \leq M$ and $\|R(0) + \lambda R''(0)\| \leq M$ when $|\lambda| \leq r$. Hence

$$\|R(0) + \lambda R^2(0)\| \leq M, \quad \|R(0) + \lambda R^3(0)\| \leq M, \quad |\lambda| \leq r. \quad (21)$$

Suppose $M_0 := \|P_0 R(0)\| < M$ (see (28)). By continuity, there exists $\delta_2 > 0$ such that

$$\psi(t) \geq \frac{M}{M_0}(1 - t) \implies t \geq t_0 + \delta_2. \quad (22)$$

There clearly exists $\psi_1 \in \Psi$ that satisfies (32) and the following condition

$$\psi_1(t) = \psi(t), \quad \forall t \in \left[t_0 + \frac{\delta_2}{2}, 1\right].$$

Let $\|\cdot\|'$ denote the norm on $\mathcal{Y}_1 := Y \oplus_{\psi_1} \mathbb{C}$. Note that \mathcal{Y} and \mathcal{Y}_1 coincide as vector spaces but are equipped with different norms $\|\cdot\|$ and $\|\cdot\|'$, which are equivalent to each other (see (27)).

Since $\|R(0)\| = M$, there exist $u_n \in \mathcal{Y}$, $n \in \mathbb{N}$ such that $\|u_n\| = \frac{1}{M}$ and $\|R(0)u_n\| \rightarrow 1$ as $n \rightarrow \infty$. Since $R(0)$ is compact, one can assume, after going to a subsequence, that $R(0)u_n$ converges to a vector $x \in \mathcal{Y}$ and $\|x\| = 1$.

Denoting for brevity $z = \|P_0x\|$, $v = |P_1x|$, we get

$$z = \lim_{n \rightarrow \infty} \|P_0R(0)u_n\| \leq \|P_0R(0)\| \|u_n\| = M_0/M$$

and

$$\begin{aligned} (z+v) \psi\left(\frac{v}{z+v}\right) &= \|x\| = 1 \implies \\ \psi\left(\frac{v}{z+v}\right) &= \frac{1}{z+v} = \frac{1}{z} \left(1 - \frac{v}{z+v}\right) \geq \frac{M}{M_0} \left(1 - \frac{v}{z+v}\right) \implies \\ \frac{v}{z+v} &\geq t_0 + \delta_2 \implies \psi_1\left(\frac{v}{z+v}\right) = \psi\left(\frac{v}{z+v}\right) \implies \\ \|x\|' &= \|x\| = 1 \end{aligned}$$

(see (26) and (22)). Also, by continuity, there exists $r_0 \in (0, r]$ such that

$$\begin{aligned} \frac{|P_1x + \lambda P_1R(0)x|}{\|P_0x + \lambda P_0R(0)x\| + |P_1x + \lambda P_1R(0)x|} &\geq \frac{v}{z+v} - \frac{\delta_2}{2} \\ &\geq t_0 + \frac{\delta_2}{2}, \quad |\lambda| \leq r_0, \end{aligned}$$

and hence $\|x + \lambda R(0)x\|' = \|x + \lambda R(0)x\|$ when $|\lambda| \leq r_0$.

Since ψ_1 satisfies (32), \mathcal{Y}_1 is complex strictly convex. Let $y := r_0 R(0)x$. Then $y \neq 0$. On the other hand, the first inequality in (21) implies

$$\begin{aligned} \|x + \zeta y\|' &= \|x + \zeta r_0 R(0)x\|' = \|x + \zeta r_0 R(0)x\| \\ &= \lim_{n \rightarrow \infty} \|R(0)u_n + \zeta r_0 R^2(0)u_n\| \leq \|R(0) + \zeta r_0 R^2(0)\| \|u_n\| \\ &\leq M \frac{1}{M} = 1, \quad |\zeta| \leq 1. \end{aligned}$$

Since $\|x\|' = \|x\| = 1$, we get a contradiction with the complex strict convexity of \mathcal{Y}_1 . Hence $\|P_0R(0)\| < M$ cannot hold.

Part 4. Since $\|P_0R(0)\| = M$, we can prove as in Part 3 that there exist $u_n \in \mathcal{Y}$, $n \in \mathbb{N}$ such that $\|u_n\| = \frac{1}{M}$ and $R(0)u_n$ converges to a vector $x \in \mathcal{Y}$ with $\|P_0x\| = 1$. Suppose $P_0R(0)x = 0$ and $P_0R^2(0)x = 0$. Then $R(0)x, R^2(0)x \in \mathbb{C}$ and there exist $\mu, \eta \in \mathbb{C}$ such that $|\mu| + |\eta| = 1$ and $\mu R(0)x + \eta R^2(0)x = 0$. Further,

$$\begin{aligned} R(0)(\mu x + \eta R(0)x) &= 0 \implies \mu x + \eta R(0)x = 0 \implies \\ \mu P_0x &= 0 \implies \mu = 0 \implies R^2(0)x = 0 \implies x = 0. \end{aligned}$$

This contradiction shows that at least one of $P_0R(0)x$ and $P_0R^2(0)x$ is nonzero.

Part 5. Suppose $P_0R^2(0)x \neq 0$ and let $y_0 := rP_0R^2(0)x$. Then the second inequality in (21) implies

$$\begin{aligned} \|P_0x + \zeta y_0\| &= \lim_{n \rightarrow \infty} \|P_0R(0)u_n + \zeta rP_0R^3(0)u_n\| \\ &\leq \|R(0) + \zeta r_0R^3(0)\| \|u_n\| \leq M \frac{1}{M} = 1, \quad |\zeta| \leq 1. \end{aligned}$$

The complex strict convexity of Y implies that $y_0 = 0$. This contradiction shows that $P_0R^2(0)x \neq 0$ cannot hold.

Part 6. Similarly, one shows that $P_0R(0)x \neq 0$ cannot hold either. Since this exhausts our list of possibilities, we conclude that there cannot exist $\lambda_0 \in \Omega$ such that $\|R(\lambda_0)\| = M$. This proves our claim in the case of ψ satisfying the conditions stated at the beginning of Part 3 above.

Part 7. Finally, suppose $\psi(t) > 1 - t$ for all $t \in (0, 1/2]$ and $\psi(t) = t$ for t sufficiently close to 1. Then $\psi^*(t) > t$ for all $t \in [1/2, 1)$ (see (34)). Hence ψ^* satisfies either the conditions in Part 1 or those in Part 3, and our claim follows by duality from what has already been proved (cf. Part 2 above and Part 2 of the proof of Theorem 2.2).

□

Theorem 3.3. *Let Ω be an open subset of \mathbb{C} , $\psi \in \Psi$, Y be a complex uniformly convex Banach space with a complex uniformly convex dual Y^* , and let $\mathcal{Y} := Y \oplus_\psi \mathbb{C}$. Suppose B is the infinitesimal generator of a C_0 semigroup on \mathcal{Y} and $B - \lambda I$ is invertible for all $\lambda \in \Omega$. If $\|(B - \lambda I)^{-1}\| \leq M$ for all $\lambda \in \Omega$, then $\|(B - \lambda I)^{-1}\| < M$ for all $\lambda \in \Omega$.*

Proof. The proof follows the same lines as that of Theorem 3.2 but it is somewhat more technical. We can assume as above that Ω is connected.

Part 1. If ψ satisfies (32), then \mathcal{Y} is complex uniformly convex and our claim follows from [34].

Part 2. Suppose there exist $t_0 \in (0, 1/2]$ and $t_1 \in [1/2, 1)$ such that $\psi(t_0) = 1 - t_0$ and $\psi(t_1) = t_1$. Then ψ^* satisfies (32) (see (33)–(34)), and $\mathcal{Y}^* = Y^* \oplus_{\psi^*} \mathbb{C}$ (see [27]) is complex uniformly convex. If \mathcal{Y} is reflexive, B^* is the infinitesimal generator of a C_0 semigroup on \mathcal{Y}^* (see, e.g., [1, Corollary 3.3.9]), and our claim follows by duality from the main result in [34] applied to B^* . If \mathcal{Y} is not reflexive, one can use the result in Remark B.4 instead of the latter (see also Theorem 2.1).

Part 3. Suppose $\psi(t) > t$ for all $t \in [1/2, 1)$ and $\psi(t) = 1 - t$ for sufficiently small $t > 0$. Then there exists $t_0 \in (0, 1/2)$ such that $\psi(t) = 1 - t$ for all $t \in [0, t_0]$ and $\psi(t) > 1 - t$ for all $t \in (t_0, 1/2]$.

Suppose there exists $\lambda_0 \in \Omega$ such that $\|R(\lambda_0)\| = M$, where $R(\lambda) := (B - \lambda I)^{-1}$. Then, exactly as in Part 3 of the proof of Theorem 3.2, one arrives at the same estimates as in (21):

$$\|R(0) + \lambda R^2(0)\| \leq M, \quad \|R(0) + \lambda R^3(0)\| \leq M, \quad |\lambda| \leq r. \quad (23)$$

Suppose $M_0 := \|P_0 R(0)\| < M$. Take $\delta_1 > 0$ such that

$$\varrho := \frac{M(1 - \delta_1)}{M_0} > 1.$$

By continuity, there exists $\delta_2 > 0$ such that

$$\psi(t) \geq \varrho(1 - t) \implies t \geq t_0 + \delta_2. \quad (24)$$

There exists $\psi_1 \in \Psi$ that satisfies (32) and the condition

$$\psi_1(t) = \psi(t), \quad \forall t \in \left[t_0 + \frac{\delta_2}{2}, 1\right].$$

Let $\|\cdot\|'$ denote the norm on $\mathcal{Y}_1 := Y \oplus_{\psi_1} \mathbb{C}$.

For any $\delta \in (0, \delta_1]$, there exists $u \in \mathcal{Y}$ such that $\|u\| = 1/M$ and $\|R(0)u\| > 1 - \delta$. Denoting for brevity $z = \|P_0 R(0)u\|$, $v = \|P_1 R(0)u\|$, we get $z \leq M_0/M$ and

$$\begin{aligned} (z + v) \psi\left(\frac{v}{z + v}\right) &> 1 - \delta \implies \\ \psi\left(\frac{v}{z + v}\right) &> \frac{1 - \delta}{z + v} = \frac{1 - \delta}{z} \left(1 - \frac{v}{z + v}\right) \\ &\geq \frac{M(1 - \delta)}{M_0} \left(1 - \frac{v}{z + v}\right) \geq \varrho \left(1 - \frac{v}{z + v}\right) \implies \\ \frac{v}{z + v} &\geq t_0 + \delta_2 \implies \psi\left(\frac{v}{z + v}\right) = \psi_1\left(\frac{v}{z + v}\right) \\ &\implies \|R(0)u\| = \|R(0)u\|'. \end{aligned}$$

Also, by continuity, there exists $r_0 \in (0, r]$ such that

$$\begin{aligned} \frac{|P_1 R(0)u + \lambda P_1 R^2(0)u|}{\|P_0 R(0)u + \lambda P_0 R^2(0)u\| + |P_1 R(0)u + \lambda P_1 R^2(0)u|} &\geq \frac{v}{z + v} - \frac{\delta_2}{2} \\ &\geq t_0 + \frac{\delta_2}{2}, \quad |\lambda| \leq r_0, \end{aligned}$$

and hence $\|R(0)u + \lambda R^2(0)u\|' = \|R(0)u + \lambda R^2(0)u\|$.

Since ψ_1 satisfies (32), \mathcal{Y}_1 is complex uniformly convex. Take an arbitrary $\tau > 0$ and consider δ corresponding to $\varepsilon := r_0\tau/2$ in the definition of complex uniform convexity. Decreasing δ if necessary, we can assume that $\delta \leq \max\{1/2, \delta_1\}$. Let $x := R(0)u$ and $y := r_0 R^2(0)u$. Then the first inequality in (23) implies

$$\begin{aligned}\|x + \zeta y\|' &= \|R(0)u + \zeta r_0 R^2(0)u\|' = \|R(0)u + \zeta r_0 R^2(0)u\| \\ &\leq \|R(0) + \zeta r_0 R^2(0)\| \|u\| \leq M \frac{1}{M} = 1, \quad |\zeta| \leq 1.\end{aligned}$$

Since $\|x\|' = \|x\| > 1 - \delta$, the complex uniform convexity of \mathcal{Y}_1 implies that $\|y\|' < \varepsilon$. Now it follows from (27) that $\|R^2(0)u\| \leq 2\|R^2(0)u\|' < \tau$, i.e. $\|B^{-2}u\| < \tau$.

Applying (37) with $w := B^{-2}u \in \text{Dom}(B^2)$ and taking into account that $\|w\| < \tau$ and

$$\|Bw\| = \|B^{-1}u\| = \|R(0)u\| > 1 - \delta \geq 1/2, \quad \|B^2w\| = \|u\| = \frac{1}{M},$$

we obtain

$$\frac{1}{4} \leq C \left(\frac{1}{M} + \tau \right) \tau,$$

where $\tau > 0$ can be taken arbitrarily small. This contradiction shows that $\|P_0 R(0)\| < M$ cannot hold, i.e. that $\|P_0 R(0)\| = M$.

Part 4. Since $\|P_0 R(0)\| = M$, for any $\delta \in (0, 1/2]$ there exists $u \in \mathcal{Y}$ such that $\|u\| = 1/M$ and $\|P_0 R(0)u\| > 1 - \delta$. Suppose $\theta(R^2(0)u) < \rho$ and $\theta(R^3(0)u) < \rho$ (see (29)), where $\rho > 0$ is a sufficiently small number to be chosen later. Since $P_1 R^2(0)u, P_1 R^3(0)u \in \mathbb{C}$, there exist $\mu, \eta \in \mathbb{C}$ such that $|\mu| + |\eta| = 1$ and $\mu P_1 R^2(0)u + \eta P_1 R^3(0)u = 0$. Then (31) implies

$$\begin{aligned}\|\mu R^2(0)u + \eta R^3(0)u\| &= \|\mu P_0 R^2(0)u + \eta P_0 R^3(0)u\| \\ &\leq 2\rho(|\mu|\|R^2(0)u\| + |\eta|\|R^3(0)u\|) \\ &\leq 2\rho(|\mu|M + |\eta|M^2) \leq 2M(M+1)\rho.\end{aligned}$$

Applying (37) with $w := \mu R^2(0)u + \eta R^3(0)u \in \text{Dom}(B^2)$, we get

$$\begin{aligned}\|\mu R(0)u + \eta R^2(0)u\|^2 &\leq C(\|\mu u + \eta R(0)u\| + 2M(M+1)\rho) 2M(M+1)\rho \\ &\leq C\left(\frac{1}{M} + 1 + 2M(M+1)\rho\right) 2M(M+1)\rho.\end{aligned}$$

Let $\mathcal{M}_1(\rho)$ denote the square root of the right-hand side of the last inequality. Then

$$\begin{aligned}\|\mu P_0 R(0)u + \eta P_0 R^2(0)u\| &\leq \|\mu R(0)u + \eta R^2(0)u\| \\ &\leq \mathcal{M}_1(\rho) = O(\sqrt{\rho}) \quad \text{as } \rho \rightarrow 0.\end{aligned}$$

Hence

$$\begin{aligned}\frac{|\mu|}{2} &\leq |\mu|(1 - \delta) \leq \|\mu P_0 R(0)u\| \leq \mathcal{M}_1(\rho) + \|P_0 R^2(0)u\| \\ &\leq \mathcal{M}_1(\rho) + 2\rho \|R^2(0)u\| \leq \mathcal{M}_1(\rho) + 2\rho M =: \mathcal{M}_2(\rho) \\ &= O(\sqrt{\rho}) \quad \text{as } \rho \rightarrow 0\end{aligned}$$

(see (31)), and $|\eta| = 1 - |\mu| \geq 1 - 2\mathcal{M}_2(\rho)$. Hence

$$\begin{aligned}(1 - 2\mathcal{M}_2(\rho)) \|R^2(0)u\| &\leq \|\eta R^2(0)u\| \leq \mathcal{M}_1(\rho) + \|\mu R(0)u\| \\ &\leq \mathcal{M}_1(\rho) + 2\mathcal{M}_2(\rho)\end{aligned}$$

and

$$\|R^2(0)u\| \leq \frac{\mathcal{M}_1(\rho) + 2\mathcal{M}_2(\rho)}{1 - 2\mathcal{M}_2(\rho)} =: \mathcal{M}_3(\rho) = O(\sqrt{\rho}) \quad \text{as } \rho \rightarrow 0.$$

Applying (37) with $w := R^2(0)u \in \text{Dom}(B^2)$, we get

$$\frac{1}{4} \leq C \left(\frac{1}{M} + \mathcal{M}_3(\rho) \right) \mathcal{M}_3(\rho).$$

There exists ρ_0 depending only on M and on C in (37) such that the above inequality fails for all $\rho \in (0, \rho_0]$. This contradiction shows that at least one of the inequalities $\theta(R^2(0)u) \geq \rho_0$ and $\theta(R^3(0)u) \geq \rho_0$ has to hold.

Part 5. Suppose $\theta(R^3(0)u) \geq \rho_0$. Here, u is such that $\|u\| = 1/M$ and $\|P_0 R(0)u\| > 1 - \delta$ with a sufficiently small $\delta \in (0, 1/2]$. Take an arbitrary $\tau > 0$ and consider δ corresponding to $\varepsilon := r\tau$ in the definition of complex uniform convexity applied to the space Y . Let $x := P_0 R(0)u$ and $y := rP_0 R^3(0)u$. Then the second inequality in (23) implies

$$\begin{aligned}\|x + \zeta y\| &= \|P_0 R(0)u + \zeta r P_0 R^3(0)u\| = \|P_0(R(0)u + \zeta r R^3(0)u)\| \\ &\leq \|R(0)u + \zeta r R^3(0)u\| \leq \|R(0) + \zeta r R^3(0)\| \|u\| \\ &\leq M \frac{1}{M} = 1, \quad |\zeta| \leq 1.\end{aligned}$$

Since $\|x\| > 1 - \delta$, the complex uniform convexity of Y implies that $\|y\| < \varepsilon$. Hence $\|P_0 R^3(0)u\| < \tau$. Since $\theta(R^3(0)u) \geq \rho_0$, it follows from (30) that

$\|R^3(0)u\| < \tau/\rho_0$. Applying Theorem B.2 with $k = 2$, $n = 3$, and $w := R^3(0)u \in \text{Dom}(B^3)$, we obtain

$$\frac{1}{8} \leq L_{3,2} \left(\frac{1}{M} + \frac{\tau}{\rho_0} \right)^2 \frac{\tau}{\rho_0},$$

where $\tau > 0$ can be taken arbitrarily small. This contradiction shows that $\theta(R^3(0)u) \geq \rho_0$ cannot hold.

Part 6. Similarly, one shows that $\theta(R^2(0)u) \geq \rho_0$ cannot hold either. Since this exhausts our list of possibilities, we conclude that there cannot exist $\lambda_0 \in \Omega$ such that $\|R(\lambda_0)\| = M$. This proves our claim in the case of ψ satisfying the conditions stated at the beginning of Part 3 above.

Part 7. Finally, suppose $\psi(t) > 1 - t$ for all $t \in (0, 1/2]$ and $\psi(t) = t$ for t sufficiently close to 1. Then $\psi^*(t) > t$ for all $t \in [1/2, 1)$ (see (34)). Hence ψ^* satisfies either the conditions in Part 1 or those in Part 3, and one can repeat the above arguments using (43) instead of (37) and (42) (cf. Part 2). \square

A Auxiliary results on geometry of Banach spaces

In this first appendix we summarize some well known concepts and theorems that are used in the paper.

Definition A.1. *A Banach space Y is called*

(i) complex strictly convex (strictly convex) if

$$x, y \in Y, \|x\| = 1 \text{ and } \|x + \zeta y\| \leq 1, \quad \forall \zeta \in \mathbb{C} \quad (\forall \zeta \in \mathbb{R}) \text{ with } |\zeta| \leq 1 \\ \text{implies } y = 0;$$

(ii) complex uniformly convex (uniformly convex) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$x, y \in Y, \|y\| \geq \varepsilon \text{ and } \|x + \zeta y\| \leq 1, \quad \forall \zeta \in \mathbb{C} \quad (\forall \zeta \in \mathbb{R}) \text{ with } |\zeta| \leq 1 \\ \text{implies } \|x\| \leq 1 - \delta.$$

It is clear that uniform convexity implies both complex uniform convexity and strict convexity, while each of these two properties implies complex strict convexity. Hilbert spaces and the L_p spaces with $1 < p < \infty$ are uniformly

convex ([12], see also [14, Ch. III, §1] or [10, Theorem 11.10]). L_1 is complex uniformly convex (see [18]) but not strictly convex. L_∞ does not have any of the above properties, but $(L_\infty)^*$ is complex uniformly convex. Indeed, this space is isometrically isomorphic to a space of bounded finitely additive set functions (see [16, Ch. IV, §8, Theorem 16 and Ch. III, §1, Lemma 5]) which is complex uniformly convex (see [29]). Hence the class of spaces to which Theorem 2.2 applies includes Hilbert spaces and $L_p(S, \Sigma, \mu)$ with $1 \leq p \leq \infty$, where (S, Σ, μ) is an arbitrary measure space.

If $1 \leq p < \infty$, then the p -direct sum $X \oplus_p Y$ of Banach spaces X and Y is the algebraic direct sum $X \oplus Y$ endowed with the norm

$$\|(x, y)\|_p = (\|x\|_X^p + \|y\|_Y^p)^{1/p}.$$

Similarly the ∞ -direct sum $X \oplus_\infty Y$ is $X \oplus Y$ with the norm

$$\|(x, y)\|_\infty = \max \{\|x\|_X, \|y\|_Y\}.$$

These definitions are special cases of the absolute norms that we describe next. Following [2], [4, §21] we say that a norm $\|\cdot\|$ on \mathbb{C}^2 is *absolute* if

$$\|(z, w)\| = \|(|z|, |w|)\|, \quad \forall (z, w) \in \mathbb{C}^2$$

and *normalized* if

$$\|(1, 0)\| = \|(0, 1)\| = 1.$$

Let N_a denote the set of all absolute normalized norms on \mathbb{C}^2 .

Let Ψ denote the set of all continuous, convex functions ψ on $[0, 1]$ such that $\psi(0) = \psi(1) = 1$ and

$$\max\{1 - t, t\} \leq \psi(t) \leq 1, \quad 0 \leq t \leq 1. \quad (25)$$

Theorem A.2. (see [4, §21]).

The formula $\|\cdot\| \rightarrow \psi(t) \equiv \|(1 - t, t)\|$ defines a one-one map from N_a onto Ψ with inverse $\psi \rightarrow \|\cdot\|_\psi$ given by

$$\|(z, v)\|_\psi := \begin{cases} (|z| + |v|) \psi\left(\frac{|v|}{|z| + |v|}\right), & (z, v) \neq (0, 0), \\ 0, & (z, v) = (0, 0). \end{cases} \quad (26)$$

If $\psi(t) \equiv \max\{1 - t, t\}$, then $\|\cdot\|_\psi$ coincides with the l_∞ norm, while if $\psi(t) \equiv 1$, then $\|\cdot\|_\psi$ coincides with the l_1 norm. Moreover

$$\frac{1}{2} \|x\|_1 \leq \|x\|_\infty \leq \|x\|_\psi \leq \|x\|_1 \leq 2 \|x\|_\infty \quad (27)$$

for all $\psi \in \Psi$ and all $x \in \mathbb{C}^2$.

Let $\psi \in \Psi$. Then the ψ -direct sum $X \oplus_\psi Y$ of the Banach spaces X and Y is the space $X \oplus Y$ equipped with the norm

$$\|(x, y)\| = \|(\|x\|_X, \|y\|_Y)\|_\psi.$$

Let

$$P_0 : X \oplus_\psi Y \rightarrow X, \quad P_1 : X \oplus_\psi Y \rightarrow Y \quad (28)$$

be the canonical projections, and let

$$\theta(u) := \frac{\|P_0 u\|_X}{\|P_0 u\|_X + \|P_1 u\|_Y} = \frac{\|x\|_X}{\|x\|_X + \|y\|_Y}, \quad u = (x, y) \in X \oplus_\psi Y. \quad (29)$$

Then

$$\begin{aligned} \|u\| &= (\|P_0 u\|_X + \|P_1 u\|_Y) \psi \left(\frac{\|P_1 u\|_Y}{\|P_0 u\|_X + \|P_1 u\|_Y} \right) \\ &= \frac{1}{\theta(u)} \|P_0 u\|_X \psi \left(\frac{\|P_1 u\|_Y}{\|P_0 u\|_X + \|P_1 u\|_Y} \right) \leq \frac{1}{\theta(u)} \|P_0 u\|_X, \end{aligned} \quad (30)$$

$$\|P_0 u\|_X = \theta(u) (\|P_0 u\|_X + \|P_1 u\|_Y) \leq 2\theta(u) \|u\| \quad (31)$$

(see (27)).

The space $X \oplus_\psi Y$ is complex uniformly (strictly) convex if and only if X and Y are complex uniformly (strictly) convex and

$$\psi(t) > \max\{1 - t, t\}, \quad \forall t \in (0, 1) \quad (32)$$

(see [15]). It is interesting to compare this result to its real valued counterpart: the space $X \oplus_\psi Y$ is uniformly (strictly) convex if and only if X and Y are uniformly (strictly) convex and ψ is strictly convex, i.e.

$$s, t \in [0, 1], \quad s \neq t, \quad 0 < c < 1 \implies \psi((1 - c)s + ct) < (1 - c)\psi(s) + c\psi(t)$$

(see [24, 31, 36]).

Suppose $\psi(t_0) = 1 - t_0$ for some $t_0 \in (0, 1/2]$. Then the equality $\psi(0) = 1$, (25) and the definition of a convex function imply that $\psi(t) = 1 - t$ for all $t \in [0, t_0]$. Similarly, if $\psi(t_1) = t_1$ for some $t_1 \in [1/2, 1)$, then $\psi(t) = t$ for all $t \in [t_1, 1]$.

The dual of $\|\cdot\|_\psi$ is the absolute normalized norm $\|\cdot\|_{\psi^*}$ with

$$\psi^*(t) := \max_{0 \leq s \leq 1} \frac{(1 - t)(1 - s) + ts}{\psi(s)} \quad (33)$$

(see [2], [27] or [22, Theorem 5.4.19]). If $\psi(s_0) = 1 - s_0$ for some $s_0 \in (0, 1/2]$, then

$$\psi^*(t) \geq \frac{(1-t)(1-s_0) + ts_0}{1-s_0} = (1-t) + \frac{s_0}{1-s_0}t > 1-t, \quad \forall t \in (0, 1].$$

Similarly, if $\psi(s_1) = s_1$ for some $s_1 \in [1/2, 1)$, then

$$\psi^*(t) > t, \quad \forall t \in [0, 1). \quad (34)$$

B Kolmogorov-Kallman-Rota type inequalities

The main result in this Appendix, Theorem B.2, is used in Part 5 of the proof of Theorem 3.3. It is an extension to generators of arbitrary C_0 semigroups of a well known estimate for generators of contraction semigroups (see [11]). Considering $B - \mu I$ with a sufficiently large $\mu \geq 0$ instead of the original operator B , one can reduce the more general case to an estimate for the generator of a bounded semigroup (see (36)). In order to derive (42) from the latter, one needs estimates for intermediate powers of B , and these are provided by Lemma B.1. Theorem B.3, which extends Theorem B.2 to the adjoints of semigroup generators, is used in Part 7 of the proof of Theorem 3.3, while the result in Remark B.4 is used in Part 2 of the proof.

We start with the following Landau-Kolmogorov type inequality for n times continuously differentiable functions $f : [0, \infty) \rightarrow \mathbb{C}$:

$$\|f^{(k)}\|_\infty^n \leq M_{n,k} \|f\|_\infty^{n-k} \|f^{(n)}\|_\infty^k, \quad (35)$$

where $n \geq 2$, $k = 1, \dots, n-1$, the constant $M_{n,k} < +\infty$ does not depend on f , and $\|\cdot\|_\infty$ denotes the sup norm (see [26, 32]).

Let B_0 be the infinitesimal generator of a C_0 semigroup $(T(t))_{t \geq 0}$ on a Banach space X such that $\|T(t)\| \leq K$, $\forall t \geq 0$. Suppose $w \in \text{Dom}(B_0^n)$. Then for any $g \in X^*$ with $\|g\|_{X^*} = 1$, the function $f(t) := g(T(t)w)$ is n times continuously differentiable and

$$f^{(m)}(t) = g(T(t)B_0^m w), \quad t \geq 0, \quad m = 1, \dots, n$$

(see, e.g., [13, Lemmata 6.1.11 and 6.1.13]). Take g such that $g(B_0^k w) =$

$\|B_0^k w\|$. Then (35) implies

$$\begin{aligned}
\|B_0^k w\|^n &= (g(B_0^k w))^n \leq \left(\sup_{t \geq 0} |g(T(t)B_0^k w)| \right)^n = \|f^{(k)}\|_\infty^n \\
&\leq M_{n,k} \|f\|_\infty^{n-k} \|f^{(n)}\|_\infty^k \\
&= M_{n,k} \left(\sup_{t \geq 0} |g(T(t)w)| \right)^{n-k} \left(\sup_{t \geq 0} |g(T(t)B_0^n w)| \right)^k \quad (36) \\
&\leq M_{n,k} \left(\sup_{t \geq 0} \|T(t)w\| \right)^{n-k} \left(\sup_{t \geq 0} \|T(t)B_0^n w\| \right)^k \\
&\leq M_{n,k} K^n \|w\|^{n-k} \|B_0^n w\|^k, \quad n \geq 2, \quad k = 1, \dots, n-1
\end{aligned}$$

(cf. [11]). The optimal values of the constants $M_{n,k}$ are discussed in [32] and [11]. In particular, if $n = 2, k = 1$, then $M_{n,k} = M_{2,1} = 4$, (35) is the Landau inequality ([28])

$$\|f'\|_\infty^2 \leq 4\|f\|_\infty \|f''\|_\infty,$$

and (36) is the Kallman-Rota inequality

$$\|B_0 w\|^2 \leq 4K^2 \|w\| \|B_0^2 w\|$$

(see [23] or [30, Chapter 1, Lemma 2.8]).

Using the Kallman-Rota inequality one can easily show that if B is the infinitesimal generator of a (possibly unbounded) C_0 semigroup on a Banach space X , then there exists a constant $C > 0$ such that

$$\|Bw\|^2 \leq C(\|B^2 w\| + \|w\|)\|w\|, \quad \forall w \in \text{Dom}(B^2) \quad (37)$$

(see [34]). The proof is an almost trivial special case of the arguments given below.

It follows from (37) that for any $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that

$$\|Bw\| \leq \varepsilon \|B^2 w\| + C(\varepsilon) \|w\|, \quad \forall w \in \text{Dom}(B^2). \quad (38)$$

Lemma B.1. *For any $n \in \mathbb{N}$, $n \geq 2$, any $k = 1, \dots, n-1$, and any $\varepsilon > 0$ there exists $C_{n,k}(\varepsilon) > 0$ such that*

$$\|B^k w\| \leq \varepsilon \|B^n w\| + C_{n,k}(\varepsilon) \|w\|, \quad \forall w \in \text{Dom}(B^n). \quad (39)$$

Proof. The proof is by induction in n . The statement holds for $n = 2$ (see (38)). Suppose it holds for n . Substituting $w = Bu$ into (39) with $k = 1$ and

using (38), we get

$$\begin{aligned}\|B^2u\| &\leq \frac{\varepsilon}{2} \|B^{n+1}u\| + C_{n,1} \left(\frac{\varepsilon}{2}\right) \|Bu\| \\ &\leq \frac{\varepsilon}{2} \|B^{n+1}u\| + C_{n,1} \left(\frac{\varepsilon}{2}\right) (\rho \|B^2u\| + C(\rho)\|u\|), \quad \forall u \in \text{Dom}(B^{n+1}).\end{aligned}$$

Taking $\rho = \frac{1}{2} C_{n,1} \left(\frac{\varepsilon}{2}\right)^{-1}$, we get

$$\|B^2u\| \leq \varepsilon \|B^{n+1}u\| + C_{n+1,2}(\varepsilon)\|u\|, \quad \forall u \in \text{Dom}(B^{n+1}) \quad (40)$$

with $C_{n+1,2}(\varepsilon) = 2C_{n,1} \left(\frac{\varepsilon}{2}\right) C \left(\frac{1}{2} C_{n,1} \left(\frac{\varepsilon}{2}\right)^{-1}\right)$. Using (38) again, and then applying (40) with $\varepsilon = 1$, we obtain

$$\begin{aligned}\|Bu\| &\leq \varepsilon \|B^2u\| + C(\varepsilon)\|u\| \leq \varepsilon \|B^{n+1}u\| + C_{n+1,1}(\varepsilon)\|u\|, \\ &\quad \forall u \in \text{Dom}(B^{n+1}),\end{aligned} \quad (41)$$

where $C_{n+1,1}(\varepsilon) = \varepsilon C_{n+1,2}(1) + C(\varepsilon)$.

Substituting $w = Bu$ into (39) again and using (41), we get

$$\begin{aligned}\|B^{k+1}u\| &\leq \frac{\varepsilon}{2} \|B^{n+1}u\| + C_{n,k} \left(\frac{\varepsilon}{2}\right) \|Bu\| \\ &\leq \frac{\varepsilon}{2} \|B^{n+1}u\| + C_{n,k} \left(\frac{\varepsilon}{2}\right) (\rho \|B^{n+1}u\| + C_{n+1,1}(\rho)\|u\|).\end{aligned}$$

Taking $\rho = \frac{\varepsilon}{2} C_{n,k} \left(\frac{\varepsilon}{2}\right)^{-1}$ and denoting $k+1 = m$, we obtain

$$\|B^m u\| \leq \varepsilon \|B^{n+1}u\| + C_{n+1,m}(\varepsilon)\|u\|, \quad \forall u \in \text{Dom}(B^{n+1}), \quad m = 3, \dots, n$$

with $C_{n+1,m}(\varepsilon) = C_{n,m-1} \left(\frac{\varepsilon}{2}\right) C_{n+1,1} \left(\frac{\varepsilon}{2} C_{n,m-1} \left(\frac{\varepsilon}{2}\right)^{-1}\right)$. Together with (40), (41) this completes the proof. \square

Theorem B.2. *Let B be the infinitesimal generator of a C_0 semigroup on a Banach space X . Then for any $n \in \mathbb{N}$, $n \geq 2$, and any $k = 1, \dots, n-1$, there exists $L_{n,k} > 0$ such that*

$$\|B^k w\|^n \leq L_{n,k} (\|B^n w\| + \|w\|)^k \|w\|^{n-k}, \quad \forall w \in \text{Dom}(B^n). \quad (42)$$

Proof. There exist constants $\mu \geq 0$ and $K \geq 1$ such that the C_0 semigroup $T(t)$ generated by $B - \mu I$ satisfies the inequality $\|T(t)\| \leq K$, $\forall t \geq 0$ (see, e.g., [30, Chapter 1, Theorem 2.2]). Increasing μ if necessary, we can assume that $\|(B - \mu I)^{-1}\| \leq 1$ (see, e.g., [21, Theorem 12.3.1]).

Applying (39) with $\varepsilon = 1$ in the penultimate inequality below, we get the following from (36) with $B_0 = B - \mu I$

$$\begin{aligned}
\|B^k w\|^n &= \left\| \sum_{j=0}^k \binom{k}{j} (B - \mu I)^{k-j} \mu^j w \right\|^n \\
&\leq \left(\sum_{j=0}^k \binom{k}{j} \mu^j \|(B - \mu I)^{k-j} w\| \right)^n \\
&\leq (k+1)^{n-1} \sum_{j=0}^k \binom{k}{j}^n \mu^{jn} \|(B - \mu I)^{k-j} w\|^n \\
&\leq (k+1)^{n-1} K^n \sum_{j=0}^k \binom{k}{j}^n \mu^{jn} M_{n,k-j} \|(B - \mu I)^n w\|^{k-j} \|w\|^{n-k+j} \\
&\leq (k+1)^{n-1} K^n \left(\sum_{j=0}^k \binom{k}{j}^n \mu^{jn} M_{n,k-j} \right) \|(B - \mu I)^n w\|^k \|w\|^{n-k} \\
&=: D_{n,k} \|(B - \mu I)^n w\|^k \|w\|^{n-k} = D_{n,k} \left\| \sum_{l=0}^n \binom{n}{l} (-\mu)^l B^{n-l} w \right\|^k \|w\|^{n-k} \\
&\leq D_{n,k} (n+1)^{k-1} \sum_{l=0}^n \binom{n}{l}^k \mu^{lk} \|B^{n-l} w\|^k \|w\|^{n-k} \\
&\leq D_{n,k} (n+1)^{k-1} \sum_{l=0}^n \binom{n}{l}^k \mu^{lk} (\|B^n w\| + C_{n,n-l}(1) \|w\|)^k \|w\|^{n-k} \\
&\leq L_{n,k} (\|B^n w\| + \|w\|)^k \|w\|^{n-k}, \quad \forall w \in \text{Dom}(B^n)
\end{aligned}$$

with a suitable constant $L_{n,k}$. □

If X is reflexive, B^* is the infinitesimal generator of a C_0 semigroup on X^* (see, e.g., [1, Corollary 3.3.9]), and (42) holds with B^* in place of B . The latter is true even if X is not reflexive, although B^* might not be densely defined and the adjoint semigroup might not be strongly continuous in this case.

Theorem B.3. *Let B be the infinitesimal generator of a C_0 semigroup on a Banach space X . Then for any $n \in \mathbb{N}$, $n \geq 2$, and any $k = 1, \dots, n-1$, there exists $L_{n,k} > 0$ such that*

$$\|(B^*)^k g\|^n \leq L_{n,k} (\|(B^*)^n g\| + \|g\|)^k \|g\|^{n-k}, \quad \forall g \in \text{Dom}((B^*)^n). \quad (43)$$

Proof. We start by proving an analogue of (36) for the adjoint B_0^* of the infinitesimal generator B_0 of a C_0 semigroup $(T(t))_{t \geq 0}$ on X such that $\|T(t)\| \leq K$, $\forall t \geq 0$. The adjoint semigroup $(T^*(t))_{t \geq 0}$ is weak* continuous, B^* is its weak* infinitesimal generator, $\|T^*(t)\| \leq K$, $\forall t \geq 0$, the function $f_x(t) := (T^*(t)g)x$ is n times continuously differentiable for any $g \in \text{Dom}((B_0^*)^n)$ and any $x \in X$ with $\|x\| = 1$, and

$$f_x^{(m)}(t) = (T^*(t)(B_0^*)^m g)x, \quad t \geq 0, \quad m = 1, \dots, n$$

(see [9, Proposition 1.4.4 and Corollary 1.4.5]). Then, as for (36), one derives from (35)

$$\begin{aligned} \|(B_0^*)^k g\|^n &= \left(\sup_{\|x\|=1} |((B_0^*)^k g)x| \right)^n \leq \left(\sup_{\|x\|=1} \sup_{t \geq 0} |(T^*(t)(B_0^*)^k g)x| \right)^n \\ &= \sup_{\|x\|=1} \|f_x^{(k)}\|_\infty^n \leq M_{n,k} \sup_{\|x\|=1} \|f_x\|_\infty^{n-k} \|f_x^{(n)}\|_\infty^k \\ &= M_{n,k} \sup_{\|x\|=1} \left(\sup_{t \geq 0} |(T^*(t)g)x| \right)^{n-k} \left(\sup_{t \geq 0} |(T^*(t)(B_0^*)^n g)x| \right)^k \quad (44) \\ &\leq M_{n,k} \left(\sup_{t \geq 0} \|T^*(t)g\| \right)^{n-k} \left(\sup_{t \geq 0} \|T^*(t)(B_0^*)^n g\| \right)^k \\ &\leq M_{n,k} K^n \|g\|^{n-k} \|(B_0^*)^n g\|^k, \quad n \geq 2, \quad k = 1, \dots, n-1. \end{aligned}$$

The proof is completed by using (44) instead of (36) and repeating the proof of Theorem B.2. □

Remark B.4. *It was shown in [34] that the resolvent norm of the infinitesimal generator of a C_0 semigroup on a Banach space cannot be constant on an open set if the underlying space is complex uniformly convex. The proof relied on estimate (37). Theorem B.3 allows one to extend the main result of [34] to the case where the dual of the underlying Banach space, rather than the space itself, is complex uniformly convex. The theorem by S. Bögli and P. Siegl ([3]) mentioned in the Introduction provides an easier way of proving this ‘dual’ result.*

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